An Optimal Portfolio of Two Securities

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Abstract. The main objective of any investor is to ensure the maximum return on investment. During the realization of this goal at least two major problems appear: the first, in which of the available assets and in what proportions investor should invest. The second problem is related to the fact that, in practice, as is well known, a higher level profitability is associated with a higher risk. Therefore, an investor can select an asset with a high yield and high risk or a more or less guaranteed low yield. Two described above selection problems constitute an problem of investment portfolio formation, which decision is given by portfolio theory, described in this paper. We study in details the portfolio of the two securities (Brusov et al 2010, 2012, 2014), which represents a more simple case, containing, however, all the main features of more common Markowitz and Tobin portfolios. It appears that when selecting anti-correlated or non-correlated securities, you can create a portfolio with the risk, lower, than risk of any of the securities of portfolio, or even zero-risk portfolio (for anti-correlated securities).

Keywords: portfolio of two securities; correlated; anti-correlated; non-correlated securities.

Оптимальный портфель из двух ценных бумаг

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Аннотация. Основная цель любого инвестора заключается в том, чтобы обеспечить максимальную отдачу от инвестиций. В ходе реализации этой цели появляются, по крайней мере, две основные проблемы. Во-первых, в какой из имеющихся активов и в каких пропорциях инвестор должен инвестировать. Вторая проблема связана с тем, что на практике, как известно, высокий уровень доходности связан с высоким риском. Таким образом, инвестор может выбрать актив с высоким выходом и высоким риском или более или менее гарантированной низкой доходностью. Два описанных случая отбора представляют собой проблему формирования инвестиционного портфеля, которая исследуется в портфельной теории, описанной в настоящей статье. Мы детально изучаем здесь портфель из двух ценных бумаг (Brusov et al 2010, 2012, 2014), что представляет собой более простой случай, однако содержащий все основные возможности более общих портфелей Марковица и Тобина. Показано, что при выборе антикоррелированных или не коррелированных ценных бумаг можно создать портфель с меньшим риском, чем риск любой из ценных бумаг портфеля, или даже нулевого риска портфеля (для антикоррелированных ценных бумаг).

Ключевые слова: портфель из двух ценных бумаг; коррелирующие бумаги; антикоррелирующие бумаги; независимые бумаги.

1. A PORTFOLIO OF TWO SECURITIES

1.1. A case of complete correlation

In a case of complete correlation:

$$\rho_{12} = \rho = 1. \tag{1}$$

For the square of the portfolio risk (variance), we have:

$$\begin{aligned}\sigma^2 &= \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2 = \\ &= \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_1\sigma_2 x_1 x_2 = \\ &= \left(\sigma_1 x_1 + \sigma_2 x_2\right)^2.\end{aligned} \tag{2}$$

Extracting the square root from both sides, we obtain for portfolio risk:

$$\sigma = \left|\sigma_1 x_1 + \sigma_2 x_2\right|. \tag{3}$$

Since all variables are nonnegative, the sign of the module can be omitted:

$$\sigma = \sigma_1 x_1 + \sigma_2 x_2 \tag{4}$$

Substituting $x_1 \to 1-t; x_2 \to t$, accounting $x_1 + x_2 = 1$, we get:

$$\sigma = \left|\sigma_1 x_1 + \sigma_2 x_2\right|. \tag{5}$$

This is the equation of the segment (*AB*), where points *A* and *B* have the following coordinates: $(\cdot)A = (\mu_1, \sigma_1)$; $(\cdot)B = (\mu_2, \sigma_2)$. t runs from 0 to 1. At $t = 0$ portfolio is at point *A*, and at $t = 1$ — at the point *B*. Thus, the admissible set of portfolios in the case of complete correlation of the securities is a segment (*AB*) (Fig. 1).

If an investor forms a portfolio of minimal risk, he must incorporate in it one type of paper that has less risk, in this case, the paper *A*, and the portfolio in this case is $X = (1,0)$. Portfolio yield (effectiveness) $\mu = \mu_1$.

With a portfolio of maximum yield, it is necessary to include in it only securities with higher income, in this case, the paper *B*, and the portfolio in this case is $X = (0,1)$. Portfolio yield $\mu = \mu_2$.

1.2. Case of complete anticorrelation

In the case of complete anti-correlation:

$$\rho_{12} = \rho = -1 \tag{6}$$

For the square of the portfolio risk (variance), we have:

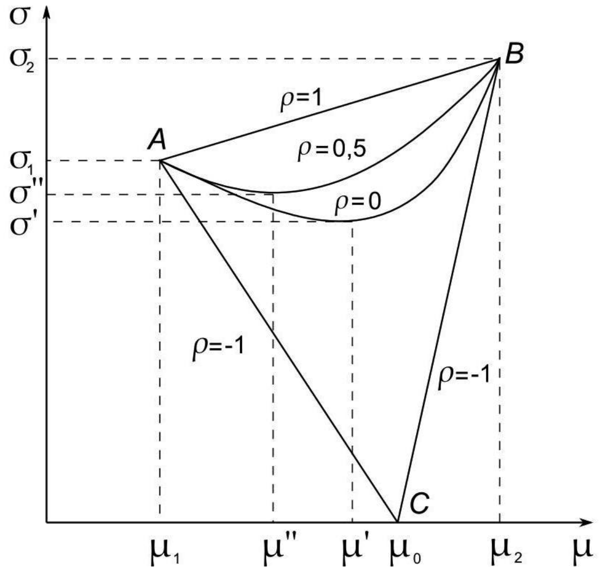

Figure 1. **The dependence of the risk of the portfolio of two securities on its effectiveness for fixed parameters of both securities and with increase in the correlation coefficient r from –1 to 1**

$$\begin{aligned}\sigma^2 &= \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2 = \\ &= \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 - 2\sigma_1\sigma_2 x_1 x_2 = \\ &= \left(\sigma_1 x_1 - \sigma_2 x_2\right)^2.\end{aligned} \tag{7}$$

Extracting the square root of both sides, we obtain for portfolio risk:

$$\sigma = \left|\sigma_1 x_1 - \sigma_2 x_2\right|. \tag{8}$$

Admissible set of portfolios in the case of complete anticorrelation of securities consists of two segments (*A*, *C*) and (*B*, *C*) (Fig. 1). In this case a risk — free portfolio (point *C*) can exists.

We find a risk — free portfolio and its profitability.

From (8) one has:

$$\sigma_1 x_1 - \sigma_2 x_2 = 0 \tag{9}$$

Substituting in (9) $x_2 = 1 - x_1$, we get:

$$\sigma_1 x_1 - \sigma_2\left(1 - x_1\right) = 0,$$

$$x_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2} \tag{10}$$

And $x_2 = 1 - x_1 = \dfrac{\sigma_2}{\sigma_1 + \sigma_2}$. (11)

Thus, free — risk portfolio has the form:

$$X = \left(\frac{\sigma_2}{\sigma_1 + \sigma_2}, \frac{\sigma_1}{\sigma_1 + \sigma_2}\right) \tag{12}$$

and its yield is equal to:

$$\mu_0 = \frac{\mu_1\sigma_2 + \mu_2\sigma_1}{\sigma_1 + \sigma_2} \tag{13}$$

Note that the risk — free portfolio does not depend on the yield of securities and is determined solely by their risks, and the pricing share of one security is proportional to the risk of another.

Since $|\rho| \le 1$, then, all admissible portfolios are located inside ($|\rho| < 1$), or on the boundary ($|\rho| = 1$), of the triangle *ABC*.

EXAMPLE 1

For a portfolio of two securities with yield and risk, respectively, $(0,2;0,5)$ and $(0,4;0,7)$ in the case of complete anticorrelation found risk — free portfolio and its profitability.

First, using the formula (4.30), we find a risk — free portfolio

$$X_0 = \left(\frac{\sigma_2}{\sigma_1 + \sigma_2}, \frac{\sigma_1}{\sigma_1 + \sigma_2}\right) = \\ = \left(\frac{0.7}{0.5 + 0.7}, \frac{0.5}{0.5 + 0.7}\right) = \left(0.583; 0.417\right).$$

Then by the formula (4.31) we find its yield

$$\mu_0 = \frac{\mu_1\sigma_2 + \mu_2\sigma_1}{\sigma_1 + \sigma_2} = \frac{0.2\cdot 0.7 + 0.4\cdot 0.5}{0.5 + 0.7} = 0.283.$$

It is seen that the portfolio yield has an intermediate value between the yields of both securities (but portfolio is risk — free!). One can check the results for portfolio yield, calculating it by the formula (4.8) $\mu = x_1\mu_1 + x_2\mu_2 = 0.583\cdot 0.2 + 0.417\cdot 0.4 = 0$.

1.3. Independent securities

For independent securities:

$$\rho_{12} = \rho = 0. \tag{14}$$

For the square of the portfolio risk (variance), we have:

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2. \tag{15}$$

Let us find a minimum — risk portfolio and its profitability and risk. For this it is necessary to minimize the objective function:

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 \quad (16)$$

under condition:

$$x_1 + x_2 = 1. \quad (17)$$

This is the task of a conditional extremum which is solved using the Lagrange function:

$$L = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \lambda\left(x_1 + x_2 - 1\right). \quad (18)$$

To find the stationary points we have the system:

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2\sigma_1^2 x_1 + \lambda = 0 \\ \dfrac{\partial L}{\partial x_2} = 2\sigma_2^2 x_2 + \lambda = 0, \\ \dfrac{\partial L}{\partial \lambda} = x_1 + x_2 - 1 = 0 \end{cases} \quad (19)$$

Subtracting the first equation from the second, we obtain:

$$\sigma_1^2 x_1 = \sigma_2^2 x_2. \quad (20)$$

Next, using the third equation, we have:

$$\sigma_1^2 x_1 = \sigma_2^2\left(1 - x_1\right). \quad (21)$$

Hence:

$$x_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2},\ x_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \quad (22)$$

Portfolio:

$$X = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right), \quad (23)$$

and its yield:

$$\mu = \frac{\mu_1 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} + \frac{\mu_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}. \quad (24)$$

The portfolio risk is equal to:

$$\sigma = \sqrt{\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2} = \sqrt{\frac{\sigma_1^2\sigma_2^4 + \sigma_1^4\sigma_2^4}{\left(\sigma_1^2 + \sigma_2^2\right)^2}} = $$
$$= \sqrt{\frac{\sigma_1^2\sigma_2^2\left(\sigma_1^2 + \sigma_2^2\right)}{\left(\sigma_1^2 + \sigma_2^2\right)^2}} = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}. \quad (25)$$

Note that in the case of three securities there is no the direct analogy with (23) (see 1.4).

EXAMPLE 2

Using formula (4.40) it is easy to demonstrate the effect of diversification on portfolio risk. Suppose a portfolio consists of two independent securities with risks $\sigma_1 = 0{,}1$ and $\sigma_2 = 0{,}2$, respectively. Let us calculate the portfolio risk by using formula (25)

$$\sigma = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \frac{0.1 \cdot 0.2}{\sqrt{0.01 + 0.04}} \approx 0.0894.$$

Thus, the portfolio risk $\sigma \approx 0.0894$ turns out to be lower than the risk of each of the securities (0.1; 0.2). This is an illustration of the principle of diversification: with "smearing" of the portfolio on an independent securities, risk is reduced.

1.4. Three independent securities

Although this case goes beyond the issue of a portfolio of two securities, we consider it here as a generalization of the case of a portfolio of two securities.

For independent securities:

$$\rho_{12} = \rho_{13} = \rho_{23} = 0\,. \quad (26)$$

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2\,. \quad (27)$$

We find a minimum — risk portfolio, its profitability and risk. For this it is necessary to minimize the objective function:

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 \quad (30)$$

under condition

$$x_1 + x_2 + x_3 = 1\,. \quad (31)$$

This is a task on conditional extremum, which is solved using the Lagrange function.

Let us form the Lagrange function and find its extremum:

$$L = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 + \lambda\left(x_1 + x_2 + x_3 - 1\right). \quad (32)$$

To find the stationary points we have the system:

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2\sigma_1^2 x_1 + \lambda = 0 \\ \dfrac{\partial L}{\partial x_2} = 2\sigma_2^2 x_2 + \lambda = 0 \\ \dfrac{\partial L}{\partial x_3} = 2\sigma_3^2 x_3 + \lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = x_1 + x_2 - 1 = 0. \end{cases} \quad (33)$$

Subtracting from the first equation the second one, then the third one, we obtain:

$$\sigma_1^2 x_1 = \sigma_2^2 x_2,$$

$$\sigma_1^2 x_1 = \sigma_2^2 x_3.$$

Hence:

$$x_2 = \frac{\sigma_1^2}{\sigma_2^2} x_1, \; x_3 = \frac{\sigma_1^2}{\sigma_3^2} x_1. \quad (34)$$

Substituting (34) in the normalization condition:

$$x_1 + x_2 + x_3 = 1. \quad (35)$$

we get:

$$x_1 + \frac{\sigma_1^2}{\sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_3^2} x_1 = 1. \quad (36)$$

Hence:

$$x_1 = \frac{1}{1 + \dfrac{\sigma_1^2}{\sigma_2^2} + \dfrac{\sigma_1^2}{\sigma_3^2}} = \frac{\sigma_2^2 \sigma_3^2}{\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2}. \quad (37)$$

Substituting this x_1 value in (34), we get two components of the portfolio:

$$x_2 = \frac{\sigma_1^2 \sigma_3^2}{\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2}. \quad (38)$$

$$x_3 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2}. \quad (39)$$

The portfolio has the form:

$$X = \frac{1}{\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2}\left(\sigma_2^2\sigma_3^2; \sigma_1^2\sigma_3^2; \sigma_1^2\sigma_2^2\right). \quad (40)$$

and its yield is equal to:

$$\mu = \frac{\mu_1\sigma_2^2\sigma_3^2 + \mu_2\sigma_1^2\sigma_3^2 + \mu_3\sigma_1^2\sigma_2^2}{\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2}. \quad (41)$$

Portfolio risk is equal to:

$$\sigma = \sqrt{\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2} = \\ = \sqrt{\frac{\left(\sigma_1^2\sigma_2^4\sigma_3^4 + \sigma_2^2\sigma_1^4\sigma_3^4 + \sigma_3^2\sigma_1^4\sigma_2^4\right)}{\left(\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2\right)^2}} = \\ = \frac{\sigma_1\sigma_2\sigma_3}{\sqrt{\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2}}. \quad (42)$$

EXAMPLE 3

For a portfolio of three independent securities with yield and risk (0.1; 0.4), (0.2; 0.6) and (0.4; 0.8) respectively, find the minimum risk portfolio, its risk and yield. Portfolio of minimum risk is given by (40)

$$X = \frac{1}{\sigma_2^2\sigma_3^2 + \sigma_1^2\sigma_3^2 + \sigma_1^2\sigma_2^2}\left(\sigma_2^2\sigma_3^2; \sigma_1^2\sigma_3^2; \sigma_1^2\sigma_2^2\right) = \\ == \frac{\left(0.6^2 \cdot 0.8^2; 0.4^2 \cdot 0.8^2; 0.4^2 \cdot 0.6^2\right)}{0.6^2 \cdot 0.8^2 + 0.4^2 \cdot 0.8^2 + 0.4^2 \cdot 0.6^2} = \\ = \frac{\left(0.2304; 0.1024; 0.0576\right)}{0.2304 + 0.1024 + 0.0576} = \\ \frac{\left(0.2304; 0.1024; 0.0576\right)}{0.3904} = \left(0.590; 0.263; 0.147\right).$$

So, $X = \left(0.590; 0.263; 0.147\right)$.

Risk of portfolio of minimum risk is found by formula (42)

$$\sigma=\frac{\sigma_1\sigma_2\sigma_3}{\sqrt{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}}=$$
$$=\frac{0.4\times0.6\times0.8}{\sqrt{0.6^2\times0.8^2+0.4^2\times0.8^2+0.4^2\times0.6^2}}=$$
$$=\frac{0.192}{\sqrt{0.2304+0.1024+0.0576}}=$$
$$=\frac{0.192}{\sqrt{0.3904}}=\frac{0.192}{0.6348}=0.307.$$

Finally, yield of portfolio of minimum risk is found by formula (41)

$$\mu=\frac{\mu_1\sigma_2^2\sigma_3^2+\mu_2\sigma_1^2\sigma_3^2+\mu_3\sigma_1^2\sigma_2^2}{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}=$$
$$=\frac{0.1\cdot0.6^2\cdot0.8^2+0.2\cdot0.4^2\cdot0.8^2+0.4\cdot0.4^2\cdot0.6^2}{0.6^2\cdot0.8^2+0.4^2\cdot0.8^2+0.4^2\cdot0.6^2}=$$
$$\frac{0.02304+0.02048+0.02304}{0.2304+0.1024+0.0576}=\frac{0.06656}{0.3904}=0.1705.$$

It is seen that the portfolio risk is less than the risk of each individual security and a portfolio yield is more than the first security yield, a little less than the yield of the second security and less than the yield of third security.

2. Risk-free security

Let one of the two portfolio securities to be risk-free. Portfolio of *n*-securities, including risk-free one, is named after Tobin, who has investigated this case for the first time, and has properties substantially different from those of the portfolio, consisting only of risky securities. Here we consider the effect of the inclusion of a risk-free securities in the portfolio of two securities.

Thus, we have two securities: $1(\mu_1, 0)$ and $2(\mu_2, \sigma_2)$, with $\mu_1 < \mu_2$ (otherwise it would be necessary to form a portfolio (1,0) consisting only of the risk — free securities, and we would have a risk — free portfolio of maximum yield).

We have the following equations:

$$\begin{gathered}\mu=\mu_1 x_1+\mu_2 x_2\\ \sigma=\sigma_2 x_2\\ x_1+x_2=1\end{gathered}\qquad(43)$$

From these equations it is easy to get an admissible set of portfolios:

$$\mu=\mu_1(1-x_2)+\mu_2 x_2=$$
$$=\mu_1+(\mu_2-\mu_1)x_2=\mu_1+(\mu_2-\mu_1)\frac{\sigma}{\sigma_2},$$

which is a segment

$$\mu=\mu_1+(\mu_2-\mu_1)\frac{\sigma}{\sigma_2},\quad 0\le\sigma\le\sigma_2\,.\qquad(44)$$

At $\sigma = 0$ portfolio is at a point 1 $(\mu_1, 0)$, and at $\sigma = \sigma_2$ at a point 2 (μ_2, σ_2) (Figure 2).

Although this case is very simple, it is nevertheless possible to draw two conclusions:

1) the admissible set of portfolios does not depend on the correlation coefficient (although usually risk-free securities considered to be uncorrelated with the other (risky) securities.

2) the admissible set of portfolios has been narrowed from a triangle to the interval.

Note that a similar effect occurs in the case of Tobin's portfolio.

In conclusion, we present the dependence of yield and risk of the portfolio on the share of the risk — free securities (Figure 3).

It is evident that the portfolio risk decreases linearly with x_1: from σ_2 at $x_1 = 0$ to zero at $x_1 = 1$, at the same time yield also decreases linearly with x_1: from μ_2 at $x_1 = 0$ to μ_1 at $x_1 = 1$.

3. Portfolio of a given yield (or given risk)

In the case of a portfolio of two securities given yield or its risk identifies portfolio uniquely (except the case $\mu_1 = \mu_2$, when only the given port-

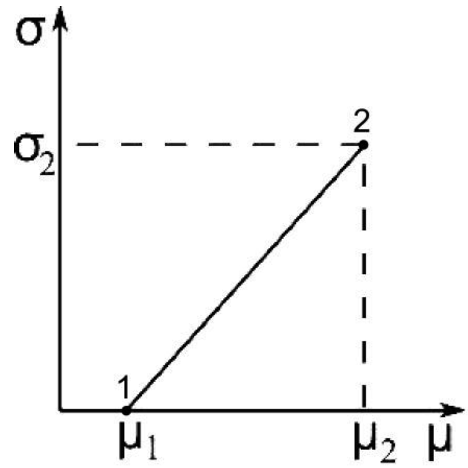

Figure 2. **Admissible set of portfolios, consisting of two securities, one of which is risk-free**

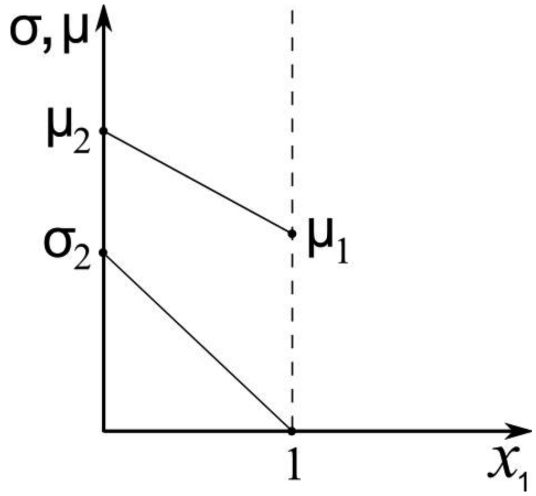

Figure 3. **Dependence of yield and risk of the portfolio on the share of the risk-free security** x_1

folio risk uniquely identifies portfolio itself, see below for details).

Under the given yield (effectiveness) of the portfolio, it is uniquely defining as the solution of the system:

$$\begin{cases} \mu = \mu_1 x_1 + \mu_2 x_2 \\ x_1 + x_2 = 1, \end{cases}. \tag{45}$$

and under the given portfolio risk, it is uniquely defining as the solution of the system:

$$\begin{cases} \sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2 \\ x_1 + x_2 = 1. \end{cases}. \tag{46}$$

Therefore, in the case of a portfolio of two securities it is not necessary to talk about the minimal boundary (minimal risk portfolio for its given effectiveness).

Let us consider the first case — the given yield of the portfolio.

We will assume that $\mu_1 \neq \mu_2$. The portfolio is uniquely defined as the solution of the system (45)

$$\begin{cases} \mu = \mu_1 x_1 + \mu_2 x_2 \\ x_1 + x_2 = 1, \end{cases}.$$

Expressing x_2 from the second equation and substituting it in the first equation, we get:

$$\mu = x_1\mu_1 + x_2\mu_2 = \\ = x_1\mu_1 + (1 - x_1)\mu_2 = x_1(\mu_1 - \mu_2) + \mu_2 .$$

Hence, we find:

$$x_1 = \frac{\mu - \mu_2}{\mu_1 - \mu_2}, \quad x_2 = \frac{\mu_1 - \mu}{\mu_1 - \mu_2}. \tag{47}$$

Substituting these expressions into the expression for the squared portfolio risk we obtain:

$$\sigma^2 = \frac{\sigma_1^2(\mu - \mu_2)^2 + \sigma_2^2(\mu - \mu_1)^2 - \\ -2\sigma_1\sigma_2\rho_{12}(\mu - \mu_1)(\mu - \mu_2)}{(\mu_2 - \mu_1)^2}. \tag{48}$$

Sometimes this equation mistakenly is called by the equation of the minimum boundary. In fact, this equation describes the connection of portfolio risk to its effectiveness.

Only at $\mu_1 = \mu_2$, when the equality $\mu = \mu_1 = \mu_2$ is valid for all the values of x_1 and x_2, and the feasible set of portfolios is narrowing from the triangle to (vertical) segment, we can speak of the minimal boundary, which in this case consists of a single point (μ, σ_1) (at $\sigma_1 < \sigma_2$) or (μ, σ_2) (at $\sigma_1 < \sigma_2$).

Let us consider different limiting cases, considered by us above.

3.1. Case of complete correlation ($\rho_{12} = 1$) and complete anti-correlation ($\rho_{12} = -1$)

As it is known, the correlation coefficient, ρ, does not exceed unity on absolute value, so let us study equation (48) for the extreme values $\rho = \pm 1$.

First, we present general considerations.

For $\rho = \pm 1$ it is known, that random variables R_1 and R_2 are linearly dependent. Without loss of generality we can assume that $R_2 = aR_1 + b$. Then, a portfolio yield can be written as follows:

$$R_X = x_1 R_1 + (1 - x_1) R_2 = \\ = (x_1 + a(1 - x_1))R_1 + (1 - x_1)b. \tag{49}$$

Therefore:

$$\sigma^2 = (x_1 + a(1 - x_1))^2 \sigma_1^2, \\ \mu = (x_1 + a(1 - x_1))\mu_1 + (1 - x_1)b. \tag{50}$$

After elimination of the parameter x_1 we obtain the following relation:

$$\sigma^2 = \left(c\mu + d\right)^2. \quad (51)$$

i.e. risk, as a function of yield will take the form of a segment or angle (Fig. 1). Now let's examine the equation (48) in cases $\rho = \pm 1$.

Case of complete correlation ($\rho_{12} = 1$ ***)***

$$\sigma = \left| \frac{\sigma_1\left(\mu - \mu_2\right) - \sigma_2\left(\mu - \mu_1\right)}{\left(\mu_2 - \mu_1\right)} \right|. \quad (52)$$

Case of complete anticorrelation ($\rho_{12} = -1$ ***)***

$$\sigma = \left| \frac{\sigma_1\left(\mu - \mu_2\right) + \sigma_2\left(\mu - \mu_1\right)}{\left(\mu_2 - \mu_1\right)} \right|. \quad (53)$$

2) **independent securities** ($\rho_{12} = 0$)
Equation (48) takes the form:

$$\sigma^2 = \frac{\sigma_1^2\left(\mu - \mu_2\right)^2 + \sigma_2^2\left(\mu - \mu_1\right)^2}{\left(\mu_2 - \mu_1\right)^2}. \quad (54)$$

Below we will show that for intermediate values of the correlation coefficient ρ portfolio risk as a function of its efficiency has the form:

$$\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}. \quad (55)$$

If one finds the shape of the dependence of risk portfolio on its effectiveness for a given portfolio $\{(\mu_1, \sigma_1), (\mu_2, \sigma_2)\}$, but for different values of the correlation coefficient, ρ, then we can come to the following conclusion: μ_M decrease when the correlation coefficient increase from −1 to 1.

In this case, a plot of the risk portfolio of its effectiveness is becoming more elongated along the horizontal axis, i.e. for a fixed change in the expected yield μ, increase in the risk σ becomes smaller (Figure 1). If we also assume that $x_1 \in [0, 1]$, and therefore $x_2 \in [0, 1]$, it is implied from the first formula (45) that $\mu \in [\mu_1, \mu_2]$ under the assumption $\mu_1 < \mu_2$ as μ is their convex combination. Portfolios are part of the boundary of AMB, namely, the part that connects the points (μ_1, σ_1) and (μ_2, σ_2) (Figure 1).

Thus, in the case $n = 2$ and under the additional assumption that $x_1 \geq 0$, $x_2 \geq 0$ the set of portfolios is a hyperbola, or pieces of broken lines connecting the points (μ_1, σ_1) and (μ_1, σ_1).

References

1. Brusov P., Brusov P.P., Orehova N., Skorodulina S. (2010). Financial mathematics for bachelor. Moscow, KNORUS, 224 p.
2. Brusov P., Brusov P.P., Orehova N., Skorodulina S. (2012). Tasks on Financial mathematics for bachelor. Moscow, KNORUS, 285 p.
3. Brusov P., Filatova T. (2014). Financial mathematics for masters. Moscow, INFRA-M, 480 p.